

# THE RICCATI SYSTEM AND A DIFFUSION-TYPE EQUATION

ERWIN SUAZO, SERGEI K. SUSLOV, AND JOSÉ M. VEGA-GUZMÁN

**ABSTRACT.** We discuss a method of constructing solution of the initial value problem for diffusion-type equations in terms of solutions of certain Riccati and Ermakov-type systems. A nonautonomous Burgers-type equation is also considered.

## 1. INTRODUCTION

A goal of this note, complementary to our recent paper [37], is to elaborate on the Cauchy initial value problem for a class of nonautonomous and inhomogeneous diffusion-type equations on  $\mathbb{R}$ . A corresponding nonautonomous Burgers-type equation is also analyzed as a by-product. Here, we use explicit transformations to the standard forms and emphasize natural relations with certain Riccati and Ermakov-type systems, which seem are missing in the available literature. Similar methods are applied to the corresponding Schrödinger equation (see, for example, [6], [7], [8], [9], [11], [24], [25], [26], [27], [36], [38], [39] and references therein). A group theoretical approach to a similar class of partial differential equations is discussed in Refs. [15], [28] and [34].

For an introduction to fundamental solutions for parabolic equations, see chapter one of the book by Friedman [14]. Among numerous applications, we only elaborate here on an important role of fundamental solutions in probability theory [10], [21]. Consider an Itô diffusion  $X = \{X_t : t \geq 0\}$  which satisfies the stochastic differential equation

$$dX_t = b(X_t, t) dt + \sigma(X_t, t) dW_t, \quad X_0 = x, \quad (1.1)$$

in which  $W = \{W_t : t \geq 0\}$  is a standard Wiener process. The existence and uniqueness of solutions of (1.1) depends on the coefficients  $b$  and  $\sigma$ . (See Ref. [21] for conditions of unique strong solution to (1.1).) If the equation (1.1) has a unique solution, then the expectations

$$u(x, t) = E_x[\phi(X_t)] = E[\phi(X_t) | X_0 = x] \quad (1.2)$$

are solutions of the Cauchy problem

$$u_t = \frac{1}{2} \sigma^2(x, t) u_{xx} + b(x, t) u_x, \quad u(x, 0) = \phi(x). \quad (1.3)$$

This PDE is known as Kolmogorov forward equation [10], [21]. Thus if  $p(x, y, t)$  is the appropriate fundamental solution of (1.3), then one can compute the given expectations according to

$$E_x[\phi(X_t)] = \int_{\Omega} p(x, y, t) \phi(y) dy. \quad (1.4)$$

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In this context, the fundamental solution is known as the probability transition density for the process and

$$\int_{\Omega} p(x, y, t) dy = 1. \quad (1.5)$$

See also Refs. [1] and [20] for applications to stochastic differential equations related to Fokker–Planck and Burgers equations.

## 2. TRANSFORMATION TO THE STANDARD FORM

We present the following result.

**Lemma 1.** *The nonautonomous and inhomogeneous diffusion-type equation*

$$\frac{\partial u}{\partial t} = a(t) \frac{\partial^2 u}{\partial x^2} - (g(t) - c(t)x) \frac{\partial u}{\partial x} + (d(t) + f(t)x - b(t)x^2) u, \quad (2.1)$$

where  $a, b, c, d, f, g$  are suitable functions of time  $t$  only, can be reduced to the standard autonomous form

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial \xi^2} \quad (2.2)$$

with the help of the following substitution:

$$u(x, t) = \frac{1}{\sqrt{\mu(t)}} e^{\alpha(t)x^2 + \delta(t)x + \kappa(t)} v(\xi, \tau), \quad (2.3)$$

$$\xi = \beta(t)x + \varepsilon(t), \quad \tau = \gamma(t).$$

Here,  $\mu, \alpha, \beta, \gamma, \delta, \varepsilon, \kappa$  are functions of  $t$  that satisfy

$$\frac{\mu'}{2\mu} + 2a\alpha + d = 0 \quad (2.4)$$

and

$$\frac{d\alpha}{dt} + b - 2c\alpha - 4a\alpha^2 = 0, \quad (2.5)$$

$$\frac{d\beta}{dt} - (c + 4a\alpha)\beta = 0, \quad (2.6)$$

$$\frac{d\gamma}{dt} - a\beta^2 = 0, \quad (2.7)$$

$$\frac{d\delta}{dt} - (c + 4a\alpha)\delta = f - 2\alpha g, \quad (2.8)$$

$$\frac{d\varepsilon}{dt} + (g - 2a\delta)\beta = 0, \quad (2.9)$$

$$\frac{d\kappa}{dt} + g\delta - a\delta^2 = 0. \quad (2.10)$$

Equation (2.5) is called the *Riccati nonlinear differential equation* [32], [40], [42] and we shall refer to the system (2.5)–(2.10) as a *Riccati-type system*.

The substitution (2.4) reduces the nonlinear Riccati equation (2.5) to the second order linear equation

$$\mu'' - \tau(t)\mu' - 4\sigma(t)\mu = 0, \quad (2.11)$$

where

$$\tau(t) = \frac{a'}{a} + 2c - 4d, \quad \sigma(t) = ab + cd - d^2 + \frac{d}{2} \left( \frac{a'}{a} - \frac{d'}{d} \right), \quad (2.12)$$

which shall be referred to as a *characteristic equation* [37].

It is also known [37] that the diffusion-type equation (2.1) has a particular solution of the form

$$u = \frac{1}{\sqrt{\mu(t)}} e^{\alpha(t)x^2 + \beta(t)xy + \gamma(t)y^2 + \delta(t)x + \varepsilon(t)y + \kappa(t)}, \quad (2.13)$$

provided that the time dependent functions  $\mu, \alpha, \beta, \gamma, \delta, \varepsilon, \kappa$  satisfy the Riccati-type system (2.4)–(2.10).

A group theoretical approach to a similar class of partial differential equations is discussed in Refs. [15], [28] and [34].

### 3. FUNDAMENTAL SOLUTION

By the *superposition principle* one can solve (formally) the Cauchy initial value problem for the diffusion-type equation (2.1) subject to initial data  $u(x, 0) = \varphi(x)$  on the entire real line  $-\infty < x < \infty$  in an integral form

$$u(x, t) = \int_{-\infty}^{\infty} K_0(x, y, t) \varphi(y) dy \quad (3.1)$$

with the *fundamental solution* (heat kernel) [37]:

$$K_0(x, y, t) = \frac{1}{\sqrt{2\pi\mu_0(t)}} e^{\alpha_0(t)x^2 + \beta_0(t)xy + \gamma_0(t)y^2 + \delta_0(t)x + \varepsilon_0(t)y + \kappa_0(t)}, \quad (3.2)$$

where a particular solution of the Riccati-type system (2.5)–(2.10) is given by

$$\alpha_0(t) = -\frac{1}{4a(t)} \frac{\mu'_0(t)}{\mu_0(t)} - \frac{d(t)}{2a(t)}, \quad (3.3)$$

$$\beta_0(t) = \frac{h(t)}{\mu_0(t)}, \quad h(t) = \exp \left( \int_0^t (c(s) - 2d(s)) ds \right), \quad (3.4)$$

$$\gamma_0(t) = \frac{d(0)}{2a(0)} - \frac{a(t)h^2(t)}{\mu_0(t)\mu'_0(t)} - 4 \int_0^t \frac{a(s)\sigma(s)h(s)}{(\mu'_0(s))^2} ds \quad (3.5)$$

$$= \frac{d(0)}{2a(0)} - \frac{1}{2\mu_1(0)} \frac{\mu_1(t)}{\mu_0(t)}, \quad (3.6)$$

$$\delta_0(t) = \frac{h(t)}{\mu_0(t)} \int_0^t \left[ \left( f(s) + \frac{d(s)}{a(s)} g(s) \right) \mu_0(s) + \frac{g(s)}{2a(s)} \mu'_0(s) \right] \frac{ds}{h(s)}, \quad (3.7)$$

$$\begin{aligned} \varepsilon_0(t) = & -\frac{2a(t)h(t)}{\mu'_0(t)} \delta_0(t) - 8 \int_0^t \frac{a(s)\sigma(s)h(s)}{(\mu'_0(s))^2} (\mu_0(s)\delta_0(s)) ds \\ & + 2 \int_0^t \frac{a(s)h(s)}{\mu'_0(s)} \left[ f(s) + \frac{d(s)}{a(s)} g(s) \right] ds, \end{aligned} \quad (3.8)$$

$$\begin{aligned} \kappa_0(t) = & -\frac{a(t)\mu_0(t)}{\mu_0'(t)}\delta_0^2(t) - 4\int_0^t \frac{a(s)\sigma(s)}{(\mu_0'(s))^2}(\mu_0(s)\delta_0(s))^2 ds \\ & + 2\int_0^t \frac{a(s)}{\mu_0'(s)}(\mu_0(s)\delta_0(s))\left[f(s) + \frac{d(s)}{a(s)}g(s)\right] ds \end{aligned} \quad (3.9)$$

with  $\delta(0) = g(0)/(2a(0))$ ,  $\varepsilon(0) = -\delta(0)$ ,  $\kappa(0) = 0$ . Here,  $\mu_0$  and  $\mu_1$  are the so-called *standard solutions* of the characteristic equation (2.11) subject to the following initial data

$$\mu_0(0) = 0, \quad \mu_0'(0) = 2a(0) \neq 0 \quad \mu_1(0) \neq 0, \quad \mu_1'(0) = 0. \quad (3.10)$$

Solution (3.3)–(3.9) shall be referred to as a *fundamental solution* of the Riccati-type system (2.5)–(2.10); see (3.27)–(3.31) and (3.32) for the corresponding asymptotics.

**Lemma 2.** *The Riccati-type system (2.4)–(2.10) has the following (general) solution:*

$$\mu(t) = -2\mu(0)\mu_0(t)(\alpha(0) + \gamma_0(t)), \quad (3.11)$$

$$\alpha(t) = \alpha_0(t) - \frac{\beta_0^2(t)}{4(\alpha(0) + \gamma_0(t))}, \quad (3.12)$$

$$\beta(t) = -\frac{\beta(0)\beta_0(t)}{2(\alpha(0) + \gamma_0(t))}, \quad (3.13)$$

$$\gamma(t) = \gamma(0) - \frac{\beta^2(0)}{4(\alpha(0) + \gamma_0(t))} \quad (3.14)$$

and

$$\delta(t) = \delta_0(t) - \frac{\beta_0(t)(\delta(0) + \varepsilon_0(t))}{2(\alpha(0) + \gamma_0(t))}, \quad (3.15)$$

$$\varepsilon(t) = \varepsilon(0) - \frac{\beta(0)(\delta(0) + \varepsilon_0(t))}{2(\alpha(0) + \gamma_0(t))}, \quad (3.16)$$

$$\kappa(t) = \kappa(0) + \kappa_0(t) - \frac{(\delta(0) + \varepsilon_0(t))^2}{4(\alpha(0) + \gamma_0(t))} \quad (3.17)$$

in terms of the fundamental solution (3.3)–(3.9) subject to arbitrary initial data  $\mu(0)$ ,  $\alpha(0)$ ,  $\beta(0)$ ,  $\gamma(0)$ ,  $\delta(0)$ ,  $\varepsilon(0)$ ,  $\kappa(0)$ .

*Proof.* Use (2.13)–(3.2), uniqueness of the solution and the elementary integral:

$$\int_{-\infty}^{\infty} e^{-ay^2+2by} dy = \sqrt{\frac{\pi}{a}} e^{b^2/a}, \quad a > 0. \quad (3.18)$$

Computational details are left to the reader.  $\square$

**Remark 1.** *It is worth noting that our transformation (2.3), combined with the standard heat kernel [29]:*

$$K_0(\xi, \eta, \tau) = \frac{1}{\sqrt{4\pi(\tau - \tau_0)}} \exp\left[-\frac{(\xi - \eta)^2}{4(\tau - \tau_0)}\right] \quad (3.19)$$

for the diffusion equation (2.2) and (3.11)–(3.17), allows one to derive the fundamental solution (3.2) of the diffusion-type equation (2.1) from a new perspective.

**Lemma 3.** *Solution (3.11)–(3.17) implies:*

$$\mu_0 = \frac{2\mu}{\mu(0)\beta^2(0)}(\gamma - \gamma(0)), \quad (3.20)$$

$$\alpha_0 = \alpha_0(t) - \frac{\beta^2}{4(\gamma - \gamma(0))}, \quad (3.21)$$

$$\beta_0 = \frac{\beta(0)\beta}{2(\gamma - \gamma(0))}, \quad (3.22)$$

$$\gamma_0 = -\alpha(0) - \frac{\beta^2(0)}{4(\gamma - \gamma(0))} \quad (3.23)$$

and

$$\delta_0 = \delta - \frac{\beta(\varepsilon - \varepsilon(0))}{2(\gamma - \gamma(0))}, \quad (3.24)$$

$$\varepsilon_0 = -\delta(0) + \frac{\beta(0)(\varepsilon - \varepsilon(0))}{2(\gamma - \gamma(0))}, \quad (3.25)$$

$$\kappa_0 = \kappa - \kappa(0) - \frac{(\varepsilon - \varepsilon(0))^2}{4(\gamma - \gamma(0))}, \quad (3.26)$$

which gives the following asymptotics

$$\alpha_0(t) = -\frac{1}{4a(0)t} - \frac{c(0)}{4a(0)} + \frac{a'(0)}{8a^2(0)} + \mathcal{O}(t), \quad (3.27)$$

$$\beta_0(t) = \frac{1}{2a(0)t} - \frac{a'(0)}{4a^2(0)} + \mathcal{O}(t), \quad (3.28)$$

$$\gamma_0(t) = -\frac{1}{4a(0)t} + \frac{c(0)}{4a(0)} + \frac{a'(0)}{8a^2(0)} + \mathcal{O}(t), \quad (3.29)$$

$$\delta_0(t) = \frac{g(0)}{2a(0)} + \mathcal{O}(t), \quad \varepsilon_0(t) = -\frac{g(0)}{2a(0)} + \mathcal{O}(t), \quad (3.30)$$

$$\kappa_0(t) = \mathcal{O}(t) \quad (3.31)$$

as  $t \rightarrow 0^+$ .

(The proof is left to the reader.)

These formulas allows to establish a required asymptotic of the fundamental solution (3.2):

$$\begin{aligned} K_0(x, y, t) &\sim \frac{1}{\sqrt{4\pi a(0)t}} \exp\left[-\frac{(x-y)^2}{4a(0)t}\right] \\ &\times \exp\left[\frac{a'(0)}{8a^2(0)}(x-y)^2 - \frac{c(0)}{4a(0)}(x^2 - y^2)\right] \exp\left[\frac{g(0)}{2a(0)}(x-y)\right]. \end{aligned} \quad (3.32)$$

(Here,  $f \sim g$  as  $t \rightarrow 0^+$ , if  $\lim_{t \rightarrow 0^+} (f/g) = 1$ . The proof is left to the reader.)

By a direct substitution one can verify that the right hand sides of (3.11)–(3.17) satisfy the Riccati-type system (2.4)–(2.10) and that the asymptotics (3.27)–(3.31) result in the continuity

with respect to initial data:

$$\lim_{t \rightarrow 0^+} \mu(t) = \mu(0), \quad \lim_{t \rightarrow 0^+} \alpha(t) = \alpha(0), \quad \text{etc.} \quad (3.33)$$

The transformation property (3.11)–(3.17) allows one to find solution of the initial value problem in terms of the fundamental solution (3.3)–(3.9) and may be referred to as a *nonlinear superposition principle* for the Riccati-type system.

#### 4. EIGENFUNCTION EXPANSION AND ERMAKOV-TYPE SYSTEM

With the help of transformation (2.3) one can reduce the diffusion equation (2.1) to another convenient form

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial \xi^2} + \xi^2 v, \quad (4.1)$$

which allows to find solution of the Cauchy initial value problem in terms of an eigenfunction expansion similar to the case of the corresponding Schrödinger in Refs. [24] and [38]. This method requires an extension the Riccati-type system (2.5)–(2.10) to a more general Ermakov-type system [24], which is integrable in quadratures once again in terms of solutions of the characteristic equation (2.11). Further details are left to the reader.

#### 5. NONAUTONOMOUS BURGERS EQUATION

The nonlinear equation

$$\begin{aligned} \frac{\partial v}{\partial t} + a(t) \left( v \frac{\partial v}{\partial x} - \frac{\partial^2 v}{\partial x^2} \right) - c(t) \left( x \frac{\partial v}{\partial x} + v \right) + g(t) \frac{\partial v}{\partial x} \\ = 2(2b(t)x - f(t)), \end{aligned} \quad (5.1)$$

when  $a = 1$  and  $b = c = f = g = 0$ , is known as *Burgers' equation* [2], [3], [5], [17], [19], [35], [41] and we shall refer to (5.1) as a nonautonomous *Burgers-type equation*.

**Lemma 4.** *The following identity holds*

$$\begin{aligned} v_t + a(vv_x - v_{xx}) + (g - cx)v_x - cv + 2(f - 2bx) \\ = -2 \left( \frac{u_t - Qu}{u} \right)_x, \end{aligned} \quad (5.2)$$

if

$$v = -2 \frac{u_x}{u} \quad (\text{The Cole-Hopf transformation}) \quad (5.3)$$

and

$$Qu = au_{xx} - (g - cx)u_x + (d + fx - bx^2)u \quad (5.4)$$

( $a, b, c, d, f, g$  are functions of  $t$  only).

(This can be verified by a direct substitution.)

The substitution (5.3) turns the nonlinear Burgers-type equation (5.1) into the diffusion-type equation (2.1). Then solution of the corresponding Cauchy initial value problem can be represented as

$$v(x, t) = -2 \frac{\partial}{\partial x} \ln \left[ \int_{-\infty}^{\infty} K_0(x, y, t) \exp \left( -\frac{1}{2} \int_0^y v(z, 0) dz \right) dy \right], \quad (5.5)$$

where the heat kernel is given by (3.2), for suitable initial data  $v(z, 0)$  on  $\mathbb{R}$ .

## 6. TRAVELING WAVE SOLUTIONS OF BURGERS-TYPE EQUATION

Looking for solutions of our equation (5.1) in the form

$$v = \beta(t) F(\beta(t)x + \gamma(t)) = \beta F(z), \quad z = \beta x + \gamma \quad (6.1)$$

( $\beta$  and  $\gamma$  are functions of  $t$  only), one gets

$$F'' = (c_0 + c_1) F' + FF' + 2c_2 z + c_3 \quad (6.2)$$

provided that

$$\beta' = c\beta, \quad \gamma' = c_0 a \beta^2, \quad (6.3)$$

$$g = c_1 a \beta, \quad b = -\frac{1}{2} c_2 a \beta^4, \quad (6.4)$$

$$f = \frac{1}{2} a \beta^3 (2c_2 \gamma + c_3) \quad (6.5)$$

( $c_0, c_1, c_2, c_3$  are constants). From (6.2):

$$F' = (c_0 + c_1) F + \frac{1}{2} F^2 + c_2 z^2 + c_3 z + c_4, \quad (6.6)$$

where  $c_4$  is a constant of integration. The substitution

$$F = -2 \frac{\mu'}{\mu} \quad (6.7)$$

transforms the Riccati equation (6.6) into a special case of generalized equation of hypergeometric type:

$$\mu'' - (c_0 + c_1) \mu' + \frac{1}{2} (c_2 z^2 + c_3 z + c_4) \mu = 0, \quad (6.8)$$

which can be solved in general by methods of Ref. [30]. Elementary solutions are discussed, for example, in [22] and [23].

## 7. SOME EXAMPLES

Now we consider from a united viewpoint several elementary diffusion and Burgers equations that are important in applications.

**Example 1** For the standard *diffusion equation* on  $\mathbb{R}$ :

$$\frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2}, \quad a = \text{constant} > 0 \quad (7.1)$$

the heat kernel is given by

$$K(x, y, t) = \frac{1}{\sqrt{4\pi at}} \exp \left[ -\frac{(x - y)^2}{4at} \right], \quad t > 0. \quad (7.2)$$

(See [4], [29] and references therein for a detailed investigation of the classical one-dimensional heat equation.)

**Example 2** In mathematical description of the nerve cell a dendritic branch is typically modeled by using cylindrical *cable equation* [18]:

$$\tau \frac{\partial u}{\partial t} = \lambda^2 \frac{\partial^2 u}{\partial x^2} + u, \quad \tau = \text{constant} > 0. \quad (7.3)$$

The fundamental solution on  $\mathbb{R}$  is given by

$$K_0(x, y, t) = \frac{\sqrt{\tau} e^{t/\tau}}{\sqrt{4\pi\lambda^2 t}} \exp \left[ -\frac{\tau(x-y)^2}{4\lambda^2 t} \right], \quad t > 0. \quad (7.4)$$

(See also [16] and references therein.)

**Example 3** The fundamental solution of the *Fokker-Planck equation* [33], [43]:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial x} + u \quad (7.5)$$

on  $\mathbb{R}$  is given by [37]:

$$K_0(x, y, t) = \frac{1}{\sqrt{2\pi(1-e^{-2t})}} \exp \left[ -\frac{(x-e^{-t}y)^2}{2(1-e^{-2t})} \right], \quad t > 0. \quad (7.6)$$

Here,

$$\lim_{t \rightarrow \infty} K_0(x, y, t) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}, \quad y = \text{constant}. \quad (7.7)$$

**Example 4** Equation

$$\frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2} + (g - kx) \frac{\partial u}{\partial x}, \quad a, k > 0, \quad g \geq 0 \quad (7.8)$$

corresponds to the heat equation with linear drift when  $g = 0$  [28]. In stochastic differential equations this equation corresponds the Kolmogorov forward equation for the regular Ornstein-Uhlenbeck process [10]. The fundamental solution is given by

$$K_0(x, y, t) = \frac{\sqrt{k} e^{kt/2}}{\sqrt{4\pi a \sinh(kt)}} \times \exp \left[ -\frac{(k(xe^{-kt/2} - ye^{kt/2}) + 2g \sinh(kt/2))^2}{4ak \sinh(kt)} \right], \quad t > 0. \quad (7.9)$$

(See Refs. [10] and [37] for more details.)

**Example 5** The *viscous Burgers equation* [2], [3], [19], [23], [41]:

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = a \frac{\partial^2 v}{\partial x^2}, \quad a = \text{constant} > 0 \quad (7.10)$$

can be linearized by the *Cole-Hopf substitution* [5], [17]:

$$v = -\frac{2a}{u} \frac{\partial u}{\partial x}, \quad (7.11)$$



which turns it into the diffusion equation (7.1). Solution of the initial value problem has the form:

$$v(x, t) = -\frac{a}{\sqrt{\pi at}} \frac{\partial}{\partial x} \ln \left[ \int_{-\infty}^{\infty} \exp \left( -\frac{(x-y)^2}{4at} - \frac{1}{2a} \int_0^y v(z, 0) dz \right) dy \right] \quad (7.12)$$

for  $t > 0$  and suitable initial data on  $\mathbb{R}$ .

**Example 6** Equation (7.10) possesses a solution of the form:

$$v = F(x + Vt), \quad V = \text{constant} \quad (7.13)$$

(we follow the original Bateman paper [2] with slightly different notations), if

$$VF' + FF' = aF'', \quad (7.14)$$

or

$$(F + V)^2 \pm A^2 = 2aF', \quad (7.15)$$

where  $A$  is a positive constant. The solution is thus either

$$v + V = A \tan \left[ \frac{A(x + Vt - c)}{2a} \right] \quad (7.16)$$

or

$$\frac{A - v - V}{A + v + V} = \exp \left[ \frac{A}{a} (x + Vt - c) \right], \quad (7.17)$$

according as the  $+$  or  $-$  sign is taken. In the first case there is no definite value of  $v$  when  $a$  tends to zero, while in the second case the limiting value of  $v$  is either  $A - V$  or  $A + V$  according as  $x + Vt$  is less or greater than  $c$ . The limiting form of the solution is thus discontinuous [2].

Further examples can be found in Refs. [10], [23], [26], [28] and [37].

## 8. CONCLUSION

In this note, we have discussed connections of certain nonautonomous and inhomogeneous diffusion-type equation and Burgers equation with solutions of the Riccati and Ermakov-type systems that seem are missing in the available literature. Traveling wave solutions of the Burgers-type equations are also discussed.

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DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF PUERTO RICO, MAYAGUEZ, CALL BOX 9000, PUERTO RICO 00681–9000.

*E-mail address:* `erwin.suazo@upr.edu`

SCHOOL OF MATHEMATICAL AND STATISTICAL SCIENCES & MATHEMATICAL, COMPUTATIONAL AND MODELING SCIENCES CENTER, ARIZONA STATE UNIVERSITY, TEMPE, AZ 85287–1804, U.S.A.

*E-mail address:* `sks@asu.edu`

*URL:* `http://hahn.la.asu.edu/~suslov/index.html`

MATHEMATICAL, COMPUTATIONAL AND MODELING SCIENCES CENTER, ARIZONA STATE UNIVERSITY, TEMPE, AZ 85287–1904, U.S.A.

*E-mail address:* `jmvega@asu.edu`